

QUOTIENTS OF STRONGLY PROPER FORCINGS AND GUESSING MODELS

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ABSTRACT. We prove that a wide class of strongly proper forcing posets have quotients with strong properties. Specifically, we prove that quotients of forcing posets which have universal strongly generic conditions on a stationary set of models by certain nice regular suborders satisfy the ω_1 -approximation property. We prove that the existence of stationarily many ω_1 -guessing models in $P_{\omega_2}(H(\theta))$, for sufficiently large cardinals θ , is consistent with the continuum being arbitrarily large, solving a problem of Viale and Weiss [13].

Many consistency results in set theory involve factoring a forcing poset \mathbb{Q} over a regular suborder \mathbb{P} in a forcing extension by \mathbb{P} , and applying properties of the quotient forcing $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. We will be interested in the situation where \mathbb{Q} has strongly generic conditions for elementary substructures, and we wish the quotient $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ to have similar properties. For example, the quotient $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ having the approximation property is useful for constructing models in which there is a failure of square principles or related properties.

We introduce some variations of strongly generic conditions, including simple and universal conditions. Our main theorem regarding quotients is that if \mathbb{Q} is a forcing poset with greatest lower bounds for which there are stationarily many countable elementary substructures which have universal strongly generic conditions, and \mathbb{P} is a regular suborder of \mathbb{Q} which relates in a nice way to \mathbb{Q} , then \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ has the ω_1 -approximation property. Several variations of this theorem are given, as well as an example which shows that not all quotients of strongly proper forcings are well behaved.

Previously Weiss introduced combinatorial principles which characterize supercompactness yet also make sense for successor cardinals ([13], [14]). Of particular interest to us is the principle $\text{ISP}(\omega_2)$, which asserts the existence of stationarily many ω_1 -guessing models in $P_{\omega_1}(H(\theta))$, for sufficiently large regular cardinals θ . This principle follows from PFA and has some of same consequences, such as the failure of the approachability property on ω_1 . It follows that $\text{ISP}(\omega_2)$ implies that $2^\omega \geq \omega_2$.

Viale and Weiss [13] asked whether this principle settles the value of the continuum. We solve this problem by showing that $\text{ISP}(\omega_2)$ is consistent with 2^ω being arbitrarily large. The solution is an application of the quotient theorem described above and the second author's method of adequate set forcing ([3]).

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Section 1 provides background on regular suborders and quotients, as well as guessing models and ISP. Section 2 introduces simple universal strongly generic conditions and proves the main result on quotients. Section 3 discusses products of strongly proper forcings. Section 4 provides two variations of the main quotient theorem. Section 5 gives an example showing that not all quotients of strongly proper forcings are well behaved. Section 6 describes a strongly proper collapse using the method of adequate set forcing. Section 7 constructs a model in which $\text{ISP}(\omega_2)$ holds and 2^ω is arbitrarily large.

1. BACKGROUND

In this section we provide the background necessary for reading the paper. We assume that the reader is already familiar with the basics of proper forcing and generalized stationarity. First, we will review some well-known results about regular suborders and quotients; since these ideas are central to the paper we provide a thorough treatment. Secondly, we review the idea of a guessing model, the approximation property, and the principle $\text{ISP}(\omega_2)$.

For the remainder of the section fix a forcing poset \mathbb{Q} which has greatest lower bounds. In other words, for all compatible conditions p and q in \mathbb{Q} , a greatest lower bound $p \wedge q$ exists. A *suborder* of \mathbb{Q} is a set $\mathbb{P} \subseteq \mathbb{Q}$ ordered by $\leq_{\mathbb{P}} := \leq_{\mathbb{Q}} \cap (\mathbb{P} \times \mathbb{P})$. A suborder \mathbb{P} of \mathbb{Q} is said to be a *regular suborder* if (a) for all p and q in \mathbb{P} , if p and q are compatible in \mathbb{Q} then p and q are compatible in \mathbb{P} , and (b) if A is a maximal antichain of \mathbb{P} , then A is a maximal antichain of \mathbb{Q} .

Lemma 1.1. *Let \mathbb{P} be a regular suborder of \mathbb{Q} . Then for all $q \in \mathbb{Q}$, there is $s \in \mathbb{P}$ such that for all $t \leq s$ in \mathbb{P} , q and t are compatible in \mathbb{Q} .*

Proof. Suppose for a contradiction that there is q in \mathbb{Q} such that for all $s \in \mathbb{P}$, there is $t \leq s$ in \mathbb{P} such that t is incompatible with q in \mathbb{Q} . Let D be the set of $t \in \mathbb{P}$ such that t is incompatible with q in \mathbb{Q} . Then D is dense in \mathbb{P} . Let A be a maximal antichain of \mathbb{P} contained in D . Since \mathbb{P} is a regular suborder, A is maximal in \mathbb{Q} . Therefore q is compatible with some member of A , which contradicts the definition of D . \square

Definition 1.2. *Let \mathbb{P} be a regular suborder of \mathbb{Q} . Then $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ is a \mathbb{P} -name for the poset consisting of conditions $q \in \mathbb{Q}$ such that for all $s \in \dot{G}_{\mathbb{P}}$, q and s are compatible in \mathbb{Q} , with the same ordering as \mathbb{Q} .*

Note that if $p * \dot{q}$ is in $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$, then p and q are compatible, so $p \wedge q$ exists.

Lemma 1.3. *Let \mathbb{P} be a regular suborder of \mathbb{Q} . Let $q \in \mathbb{Q}$ and $s \in \mathbb{P}$. Then the following are equivalent:*

- (1) $s \Vdash_{\mathbb{P}} q \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$;
- (2) for all $t \leq s$ in \mathbb{P} , q and t are compatible in \mathbb{Q} .

The proof is straightforward.

Lemma 1.4. *Let \mathbb{P} be a regular suborder of \mathbb{Q} . Then \mathbb{P} forces that whenever $q \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$ and $q \leq p$, then $p \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$.*

The proof is easy.

Lemma 1.5. *Let \mathbb{P} be a regular suborder of \mathbb{Q} . If D is a dense subset of \mathbb{Q} , then \mathbb{P} forces that $D \cap (\mathbb{Q}/\dot{G}_{\mathbb{P}})$ is a dense subset of $\mathbb{Q}/\dot{G}_{\mathbb{P}}$.*

Proof. Suppose for a contradiction that $p \in \mathbb{P}$ and p forces that q is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ but q has no extension in $D \cap (\mathbb{Q}/\dot{G}_{\mathbb{P}})$. Then p is compatible with q . Fix $r \leq q \wedge p$ in D . Apply Lemma 1.1 to find v in \mathbb{P} such that every extension of v in \mathbb{P} is compatible with r . In particular, v is compatible with r and hence with p . Since v and p are in \mathbb{P} and \mathbb{P} is a regular suborder, v and p are compatible in \mathbb{P} . So without loss of generality assume that $v \leq p$. By Lemma 1.3, v forces that r is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. Since $r \in D$, v forces that r is an extension of q in $D \cap (\mathbb{Q}/\dot{G}_{\mathbb{P}})$, contradicting that $v \leq p$. \square

Lemma 1.6. *Let \mathbb{P} be a regular suborder of \mathbb{Q} .*

- (1) *Suppose that H is a V -generic filter on \mathbb{Q} . Then $H \cap \mathbb{P}$ is a V -generic filter on \mathbb{P} and H is a $V[H \cap \mathbb{P}]$ -generic filter on $\mathbb{Q}/(H \cap \mathbb{P})$.*
- (2) *Suppose that G is a V -generic filter on \mathbb{P} and H is a $V[G]$ -generic filter on \mathbb{Q}/G . Then H is a V -generic filter on \mathbb{Q} , $G = H \cap \mathbb{P}$, and $V[G][H] = V[H]$.*

Proof. (1) Suppose that H is a V -generic filter on \mathbb{Q} . Since any maximal antichain of \mathbb{P} is a maximal antichain of \mathbb{Q} , $H \cap \mathbb{P}$ meets every maximal antichain of \mathbb{P} . A straightforward density argument shows that $H \cap \mathbb{P}$ is a filter. So $H \cap \mathbb{P}$ is a V -generic filter on \mathbb{P} .

Since H is a filter, every member of H is compatible in \mathbb{Q} with every member of $H \cap \mathbb{P}$. So $H \subseteq \mathbb{Q}/(H \cap \mathbb{P})$ and H is a filter on $\mathbb{Q}/(H \cap \mathbb{P})$. We will show that H meets every dense subset of $\mathbb{Q}/(H \cap \mathbb{P})$ in $V[H \cap \mathbb{P}]$.

Let \dot{D} be a \mathbb{P} -name for a dense subset of $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ and let $D := \dot{D}^{H \cap \mathbb{P}}$. We will show that $D \cap H \neq \emptyset$. Let D' be the set of conditions in \mathbb{Q} of the form $p \wedge q$, where p is in \mathbb{P} and p forces that q is in \dot{D} . Then D' is dense in \mathbb{Q} by a straightforward argument using Lemmas 1.1 and 1.3. Fix $p \wedge q$ in $D' \cap H$. Then $q \in H$, and since $p \in H \cap \mathbb{P}$, $q \in D$.

(2) By Lemma 1.4, H is closed upwards in \mathbb{Q} , so H is a filter on \mathbb{Q} . Let D be a dense subset of \mathbb{Q} in V , and we will show that $H \cap D \neq \emptyset$. By Lemma 1.5, $D \cap (\mathbb{Q}/G)$ is a dense subset of \mathbb{Q}/G . Since H is a $V[G]$ -generic filter on \mathbb{Q}/G , it meets $D \cap (\mathbb{Q}/G)$ and hence D .

To show that $V[G][H] = V[H]$, it suffices to show that $G = H \cap \mathbb{P}$. Since G and $H \cap \mathbb{P}$ are both V -generic filters on \mathbb{P} by (1), it suffices to show that every condition in G is compatible with every condition in $H \cap \mathbb{P}$. But if $q \in H \cap \mathbb{P}$, then $q \in \mathbb{Q}/G$, which implies that q is compatible in \mathbb{Q} with every condition in G . Since \mathbb{P} is a regular suborder of \mathbb{Q} , q is compatible in \mathbb{P} with every condition in G . \square

It follows that \mathbb{Q} is forcing equivalent to $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$. In fact, the function which sends a condition in $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$ of the form $p * \check{q}$ to $p \wedge q$ is a dense embedding defined on a dense subset of $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$.

The next technical lemma will be used later in the paper.

Lemma 1.7. *Let \mathbb{P} be a regular suborder of \mathbb{Q} . Suppose that $q \in \mathbb{Q}$, $p \in \mathbb{P}$, and p forces in \mathbb{P} that q is not in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. Then p and q are incompatible.*

Proof. Suppose for a contradiction that p forces that q is not in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$, but there is $r \leq p, q$ in \mathbb{Q} . Let H be a V -generic filter on \mathbb{Q} such that $r \in H$. Then p and q are in H . Since \mathbb{P} is a regular suborder of \mathbb{Q} , $H \cap \mathbb{P}$ is a V -generic filter on \mathbb{P} by Lemma 1.6. As $p \in H \cap \mathbb{P}$, q is not in $\mathbb{Q}/(H \cap \mathbb{P})$. Therefore q is incompatible in \mathbb{Q} with some member of $H \cap \mathbb{P}$. But this is impossible since $q \in H$ and H is a filter. \square

We now provide the necessary background on guessing models, the approximation property, and ISP.

Definition 1.8. *A set N is said to be ω_1 -guessing if for any set of ordinals $d \subseteq N$ such that $\sup(d) < \sup(N \cap On)$, if d satisfies that for any countable set $b \in N$, $d \cap b \in N$, then there exists $d' \in N$ such that $d = d' \cap N$.*

Definition 1.9. *Let W_1 and W_2 be transitive with $W_1 \subseteq W_2$. We say that the pair (W_1, W_2) has the ω_1 -approximation property if whenever $d \in W_2$ is a bounded subset of $W_1 \cap On$ and satisfies that $b \cap d \in W_1$ for any set $b \in W_1$ which is countable in W_1 , then $d \in W_1$.*

Lemma 1.10. *Let N be an elementary substructure of $H(\chi)$ for some uncountable cardinal χ . Then the following are equivalent:*

- (1) *N is an ω_1 -guessing model;*
- (2) *the pair (\overline{N}, V) has the ω_1 -approximation property, where \overline{N} is the transitive collapse of N .*

Proof. Let $\sigma : N \rightarrow \overline{N}$ be the transitive collapsing map.

(1 \Rightarrow 2) Assume that N is an ω_1 -guessing model. To show that (\overline{N}, V) has the ω_1 -approximation property, let d be a bounded subset of $\overline{N} \cap On$ and assume that whenever \overline{N} models that b is a countable set of ordinals, $b \cap d \in \overline{N}$. We will prove that $d \in \overline{N}$.

Let $e := \sigma^{-1}[d]$. Then e is a subset of $N \cap On$ and $\sup(e) < \sup(N \cap On)$. Let $b \in N$ be a countable set of ordinals, and we show that $e \cap b \in N$. Since $N \prec H(\chi)$, N models that b is countable. Therefore \overline{N} models that $\sigma(b)$ is countable. It follows that $d \cap \sigma(b) \in \overline{N}$. Hence $\sigma^{-1}(d \cap \sigma(b)) \in N$. Since b and $d \cap \sigma(b)$ are countable, $\sigma^{-1}(d \cap \sigma(b)) = \sigma^{-1}[d \cap \sigma(b)] = \sigma^{-1}[d] \cap \sigma^{-1}[\sigma(b)] = e \cap b$. Therefore $e \cap b$ is in N .

Since N is an ω_1 -guessing model, fix $e' \in N$ such that $e = e' \cap N$. Then $\sigma(e') \in \overline{N}$ and $\sigma(e') = \sigma[e' \cap N] = \sigma[e] = d$. It follows that d is in \overline{N} , as desired.

(2 \Rightarrow 1) Suppose that (\overline{N}, V) has the ω_1 -approximation property. Let $d \subseteq N \cap On$ be given with $\sup(d) < \sup(N \cap On)$, and assume that for any countable set $b \in N$, $d \cap b \in N$. Let $e := \sigma[d]$, which is a bounded subset of $\overline{N} \cap On$. Suppose that \overline{N} models that b is a countable set of ordinals. Then $\sigma^{-1}(b)$ is a countable set in N , so $d \cap \sigma^{-1}(b) \in N$. Since b and $d \cap \sigma^{-1}(b)$ are countable, $\sigma(d \cap \sigma^{-1}(b)) = \sigma[d \cap \sigma^{-1}(b)] = \sigma[d] \cap b = e \cap b$. Hence $e \cap b$ is in \overline{N} . By the ω_1 -approximation property, e is in \overline{N} . Hence $d' := \sigma^{-1}(e)$ is in N . But $d = d' \cap N$, as can be easily checked. \square

Definition 1.11. *We say that the principle $\text{ISP}(\omega_2)$ holds if for all sufficiently large regular uncountable cardinals χ , there are stationarily many N in $P_{\omega_2}(H(\chi))$ such that $N \cap \omega_2 \in \omega_2$, $N \prec H(\chi)$, and N is an ω_1 -guessing model.*

The principle $\text{ISP}(\omega_2)$ was introduced by Weiss ([13], [14]), in a different form which asserts the existence of an ineffable branch for any slender $P_{\omega_2}(\lambda)$ -list, for all cardinals $\lambda \geq \omega_2$. Roughly speaking, $\text{ISP}(\omega_2, \lambda)$ is a $P_{\omega_2}(\lambda)$ version of the tree property, where usual the tree property on ω_2 states that every ω_2 -tree has a cofinal branch. The equivalence of this principle with Definition 1.11 is proven in [13, Section 3]. See [13, Section 2] for the alternative definition of ISP and a discussion of the principle.

2. QUOTIENTS OF STRONGLY PROPER FORCINGS

In this section we prove our main theorem on quotients of strongly proper forcings. We introduce the idea of a simple universal strongly N -generic condition, and show that under some circumstances quotients of forcing posets which have such conditions for stationarily many N are well behaved.

The following definition was introduced by Mitchell in a slightly different form ([9]).

Definition 2.1. *Let \mathbb{Q} be a forcing poset, $q \in \mathbb{Q}$, and N a set. We say that q is a strongly (N, \mathbb{Q}) -generic condition if for any set D which is dense in the forcing poset $N \cap \mathbb{Q}$, D is predense in \mathbb{Q} below q .*

If \mathbb{Q} is understood from context, we just say that q is a strongly N -generic condition.

For a forcing poset \mathbb{Q} , let $\lambda_{\mathbb{Q}}$ denote the smallest uncountable cardinal λ such that $\mathbb{Q} \subseteq H(\lambda)$. Note that for any set N , a condition $q \in \mathbb{Q}$ is strongly N -generic iff q is strongly $(N \cap H(\lambda_{\mathbb{Q}}))$ -generic.

The next lemma was basically proven in [9, Proposition 2.15]. We include a proof for completeness.

Lemma 2.2. *Let \mathbb{Q} be a forcing poset, $q \in \mathbb{Q}$, and N a set. Then the following are equivalent:*

- (1) *q is strongly N -generic;*
- (2) *there is a function $r \mapsto r \restriction N$ defined on conditions $r \leq q$ satisfying that $r \restriction N \in N \cap \mathbb{Q}$ and for all $v \leq r \restriction N$ in $N \cap \mathbb{Q}$, r and v are compatible.*

Proof. For the forward direction, suppose that there is $r \leq q$ for which there does not exist a condition $r \restriction N$ all of whose extensions in $N \cap \mathbb{Q}$ are compatible with r . Let D be the set of $w \in N \cap \mathbb{Q}$ which are incompatible with r . The assumption on r implies that D is dense in $N \cap \mathbb{Q}$. But D is not predense below q since every condition in D is incompatible with r . So q is not strongly N -generic.

Conversely, assume that there is a function $r \mapsto r \restriction N$ as described. Let D be dense in $N \cap \mathbb{Q}$ and let $r \leq q$. Fix $v \leq r \restriction N$ in D . Then r and v are compatible. So D is predense below q . \square

We introduce two strengthenings of strong genericity, namely simple and universal.

Definition 2.3. *Let \mathbb{Q} be a forcing poset, $q \in \mathbb{Q}$, and N a set. We say that q is a universal strongly (N, \mathbb{Q}) -generic condition if q is a strongly (N, \mathbb{Q}) -generic condition and for all $p \in N \cap \mathbb{Q}$, p and q are compatible.*

Definition 2.4. *Let \mathbb{Q} be a forcing poset, $q \in \mathbb{Q}$, and N a set. We say that q is a simple strongly (N, \mathbb{Q}) -generic condition if there exists a set $E \subseteq \mathbb{Q}$ which is dense below q and a function $r \mapsto r \restriction N$ defined on E such that for all $r \in E$, $r \restriction N \in N \cap \mathbb{Q}$, $r \leq r \restriction N$, and for all $v \leq r \restriction N$ in $N \cap \mathbb{Q}$, r and v are compatible.*

The difference between simple and ordinary strongly generic conditions as described in Lemma 2.2(2) is the additional assumption in simple that $r \leq r \restriction N$. But if the forcing poset has greatest lower bounds, then the two ideas are equivalent; see Lemma 2.5 below.

Note that if q is a simple strongly N -generic condition, then q is strongly N -generic. Namely, given $r \leq q$ not in E , first extend r to s in E and then define $r \restriction N$ to be $s \restriction N$.

While these definitions are new, most strongly generic conditions in the literature satisfy them. For example, all of the adequate set type forcings described in [3] have simple universal strongly generic conditions for countable elementary substructures.

It turns out that for forcing posets with greatest lower bounds, every strongly generic condition is simple.

Lemma 2.5. *Let \mathbb{Q} be a forcing poset with greatest lower bounds. Let N be a set and $q \in \mathbb{Q}$. If q is a strongly N -generic condition, then q is a simple strongly N -generic condition.*

Proof. Let $r_0 \leq q$, and we will find $r \leq r_0$ and $r \restriction N \in N \cap \mathbb{Q}$ such that $r \leq r \restriction N$, and for all $v \leq r \restriction N$ in $N \cap \mathbb{Q}$, r and v are compatible. Since q is strongly N -generic, there is a condition $u \in N \cap \mathbb{Q}$ such that for all $v \leq u$ in $N \cap \mathbb{Q}$, r_0 and v are compatible. In particular, r_0 and u are compatible. Let $r := r_0 \wedge u$ and $r \restriction N := u$. Then $r \leq r_0$, $r \restriction N \in N \cap \mathbb{Q}$, and $r \leq r \restriction N$.

Let $v \leq r \restriction N$ be in $N \cap \mathbb{Q}$, and we will show that r and v are compatible. Then $v \leq u$, so by the choice of u , r_0 and v are compatible. Therefore $r_0 \wedge v$ exists. But $r_0 \wedge v \leq v$, and since $v \leq u$, $r_0 \wedge v \leq r_0 \wedge u = r$. So r and v are compatible. \square

Definition 2.6. *A forcing poset \mathbb{Q} is strongly proper if for all large enough regular cardinals χ , there are club many countable elementary substructures N of $H(\chi)$ satisfying that whenever $p \in N \cap \mathbb{Q}$, there exists a strongly N -generic condition below p .*

This definition, introduced by Mitchell, is defined in a way similar to the usual definition of proper forcing. But by standard arguments and the comments after Definition 2.1, this definition is equivalent to the existence of club many countable sets N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that every $p \in N \cap \mathbb{Q}$ has a strongly N -generic extension.

Definition 2.7. *Let \mathbb{Q} be a forcing poset. We say that \mathbb{Q} is strongly proper on a stationary set if there are stationarily many N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that whenever $p \in N \cap \mathbb{Q}$, there is $q \leq p$ which is a strongly N -generic condition.*

By standard arguments we get an equivalent property by replacing $\lambda_{\mathbb{Q}}$ with any regular cardinal $\lambda \geq \lambda_{\mathbb{Q}}$; similar comments apply to the following definition.

Definition 2.8. *A forcing poset \mathbb{Q} is said to have universal strongly generic conditions on a stationary set if there are stationarily many N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that there exists a universal strongly N -generic condition.*

For a forcing poset \mathbb{Q} with greatest lower bounds, we will consider suborders \mathbb{P} of \mathbb{Q} satisfying the following compatibility property. This property will be crucial for the rest of the paper.

$*(\mathbb{P}, \mathbb{Q})$: for all x in \mathbb{P} and y, z in \mathbb{Q} , if x, y , and z are pairwise compatible, then x is compatible with $y \wedge z$.

Note that $*(\mathbb{Q}, \mathbb{Q})$ implies that $*(\mathbb{P}, \mathbb{Q})$ for all suborders \mathbb{P} of \mathbb{Q} . A large number of forcing posets \mathbb{Q} satisfy the property $*(\mathbb{Q}, \mathbb{Q})$. These include Cohen forcings, collapsing forcings, adding a club by initial segments, and many side condition forcings. In general, forcing posets where greatest lower bounds are given by unions

tend to satisfy it. In contrast, most Boolean algebras do not satisfy the property. For example, consider a field of sets, and let A and B be sets with nonempty intersection and relative complements. Then A , B , and $A \triangle B$ have pairwise nonempty intersections, but A has empty intersection with $B \cap (A \triangle B) = B \setminus A$.

Our main theorem on quotients Theorem 2.11 states that if \mathbb{Q} has universal strongly generic conditions on a stationary set, and \mathbb{P} is a regular suborder satisfying $*(\mathbb{P}, \mathbb{Q})$, then \mathbb{P} forces that the quotient $\mathbb{Q}/G_{\mathbb{P}}$ has universal strongly generic conditions on a stationary set.

Lemma 2.9. *Let \mathbb{Q} be a forcing poset which has greatest lower bounds, and let \mathbb{P} be a regular suborder of \mathbb{Q} which satisfies $*(\mathbb{P}, \mathbb{Q})$. Then \mathbb{P} forces that for all r and s in $\mathbb{Q}/G_{\mathbb{P}}$, if r and s are compatible in \mathbb{Q} then $r \wedge s$ is in $\mathbb{Q}/G_{\mathbb{P}}$.*

Proof. Let G be a V -generic filter on \mathbb{P} , and suppose that r and s are in \mathbb{Q}/G and are compatible in \mathbb{Q} . To show that $r \wedge s$ is in \mathbb{Q}/G , let p be in G and we will show that p is compatible with $r \wedge s$. Since r and s are in \mathbb{Q}/G , they are each compatible with p . By property $*(\mathbb{P}, \mathbb{Q})$, p is compatible with $r \wedge s$. \square

The next lemma will be used in the proof of Theorem 2.11, which is our main theorem on quotients. We will use it again in Section 4 when we prove a variation of the main theorem.

Lemma 2.10. *Let \mathbb{Q} be a forcing poset which has greatest lower bounds, and let \mathbb{P} be a regular suborder of \mathbb{Q} satisfying property $*(\mathbb{P}, \mathbb{Q})$. Let χ be a regular cardinal, $N \prec H(\chi)$, $\mathbb{P}, \mathbb{Q} \in N$, and suppose that q is a strongly (N, \mathbb{Q}) -generic condition. Assume that z is in \mathbb{P} and z forces that \dot{q} is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. Then z forces that $N[\dot{G}_{\mathbb{P}}] \cap V \subseteq N$ and \dot{q} is a strongly $(N[\dot{G}_{\mathbb{P}}], \mathbb{Q}/\dot{G}_{\mathbb{P}})$ -generic condition.*

Proof. To show that z forces that $N[\dot{G}_{\mathbb{P}}] \cap V \subseteq N$, assume for a contradiction that $z' \leq z$ in \mathbb{P} , \dot{a} is a \mathbb{P} -name in N , and z' forces that \dot{a} is in $V \setminus N$. Since $\dot{a}^G = \dot{a}^{G \cap \mathbb{P}}$ whenever G is a V -generic filter on \mathbb{Q} , it follows that z' forces in \mathbb{Q} that \dot{a} is in $V \setminus N$. Let D be the dense set of conditions $v \in \mathbb{Q}$ such that v decides in \mathbb{Q} whether \dot{a} is in V , and if it decides that it is, it decides the value of \dot{a} . Since $\dot{a} \in N$, it follows by elementarity that $D \in N$, and hence $D \cap N$ is dense in $N \cap \mathbb{Q}$.

Since z forces that \dot{q} is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$, z' is compatible with q in \mathbb{Q} . As q is strongly (N, \mathbb{Q}) -generic and $q \wedge z' \leq q$, fix $v \in D \cap N$ which is compatible with $q \wedge z'$. As z' forces that \dot{a} is in $V \setminus N$, so does $v \wedge q \wedge z'$. Since $v \in D \cap N$, by the elementarity of N there is $b \in N$ such that v forces that $\dot{a} = \dot{b}$. But then $v \wedge q \wedge z'$ forces that $\dot{a} \in N$, which is a contradiction.

Now let $G_{\mathbb{P}}$ be a V -generic filter on \mathbb{P} which contains z . We will show that in $V[G_{\mathbb{P}}]$, q is a strongly $(N[G_{\mathbb{P}}], \mathbb{Q}/G_{\mathbb{P}})$ -generic condition. Since q is a strongly (N, \mathbb{Q}) -generic condition in V , by Lemma 2.5 fix a set $E \subseteq \mathbb{Q}$ in V which is dense below q and a function $r \mapsto r \restriction N$ defined on E satisfying that for all $r \in E$, $r \restriction N \in N \cap \mathbb{Q}$, $r \leq r \restriction N$, and for all $v \leq r \restriction N$ in $N \cap \mathbb{Q}$, r and v are compatible. By Lemma 1.5, $E \cap (\mathbb{Q}/G_{\mathbb{P}})$ is dense below q in $\mathbb{Q}/G_{\mathbb{P}}$.

In $V[G_{\mathbb{P}}]$ define a map $r \mapsto r \restriction N[G_{\mathbb{P}}]$ on the set $E \cap (\mathbb{Q}/G_{\mathbb{P}})$ by letting $r \restriction N[G_{\mathbb{P}}] := r \restriction N$. For all $r \in E \cap (\mathbb{Q}/G_{\mathbb{P}})$, $r \restriction N[G_{\mathbb{P}}]$ is in N and hence in $N[G_{\mathbb{P}}]$, and $r \leq r \restriction N = r \restriction N[G_{\mathbb{P}}]$. By Lemma 1.4, $r \restriction N[G_{\mathbb{P}}]$ is in $\mathbb{Q}/G_{\mathbb{P}}$. Consider $v \in N[G_{\mathbb{P}}] \cap (\mathbb{Q}/G_{\mathbb{P}})$ below $r \restriction N[G_{\mathbb{P}}]$. Since $N[G_{\mathbb{P}}] \cap V \subseteq N$ as shown above, $v \in N \cap \mathbb{Q}$ and $v \leq r \restriction N$ in \mathbb{Q} . So v is compatible with r in \mathbb{Q} . As r and v are

both in $\mathbb{Q}/G_{\mathbb{P}}$, $r \wedge v$ is in $\mathbb{Q}/G_{\mathbb{P}}$ by Lemma 2.9. Hence r and v are compatible in $\mathbb{Q}/G_{\mathbb{P}}$. \square

Theorem 2.11. *Let \mathbb{Q} be a forcing poset which has greatest lower bounds, and let \mathbb{P} be a regular suborder of \mathbb{Q} satisfying property $\ast(\mathbb{P}, \mathbb{Q})$. Assume that \mathbb{Q} has universal strongly generic conditions on a stationary set. Then \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ has universal strongly generic conditions on a stationary set. In particular, \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ is strongly proper on a stationary set.*

Proof. Let χ be a regular cardinal such that $\mathbb{Q} \in H(\chi)$. Then \mathbb{Q} forces that χ is regular and $\lambda_{\mathbb{Q}/\dot{G}_{\mathbb{P}}} \leq \chi$. Let \dot{F} be a \mathbb{P} -name for a function $\dot{F} : (H(\chi)^{V[\dot{G}_{\mathbb{P}}]})^{<\omega} \rightarrow H(\chi)^{V[\dot{G}_{\mathbb{P}}]}$, and let $s \in \mathbb{P}$. We will find an extension of s in \mathbb{P} which forces that there exists a countable set $M \subseteq H(\chi)^{V[\dot{G}_{\mathbb{P}}]}$ which is closed under \dot{F} such that there exists a universal strongly $(M, \mathbb{Q}/\dot{G}_{\mathbb{P}})$ -generic condition.

Define $H : H(\chi)^{<\omega} \rightarrow H(\chi)$ by letting $H(\dot{a}_0, \dots, \dot{a}_n)$ be a \mathbb{P} -name in $H(\chi)$ which \mathbb{P} forces is equal to $\dot{F}(\dot{a}_0, \dots, \dot{a}_n)$, for any \mathbb{P} -names $\dot{a}_0, \dots, \dot{a}_n$ in $H(\chi)$. Since \mathbb{Q} has universal strongly generic conditions on a stationary set, we can fix a countable set $N \prec H(\chi)$ such that \mathbb{P} , \mathbb{Q} , and s are in N , N is closed under H , and there is a universal strongly (N, \mathbb{Q}) -generic condition q_N .

Since $s \in N \cap \mathbb{Q}$ and q_N is universal, q_N is compatible with s . As \mathbb{P} is a regular suborder of \mathbb{Q} , fix z in \mathbb{P} such that every extension of z in \mathbb{P} is compatible with $q_N \wedge s$ in \mathbb{Q} . By Lemma 1.3, z forces that $q_N \wedge s$ is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. By Lemma 1.4, z forces that q_N is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. Since z is compatible with $q_N \wedge s$ in \mathbb{Q} , z is compatible with s in \mathbb{Q} . As \mathbb{P} is a regular suborder of \mathbb{Q} and s and z are in \mathbb{P} , z is compatible with s in \mathbb{P} . Extending z if necessary in \mathbb{P} , we may assume without loss of generality that $z \leq s$.

Since N is closed under H , \mathbb{P} forces that $N[\dot{G}_{\mathbb{P}}]$ is closed under \dot{F} . So it suffices to show that z forces that q_N is a universal strongly $(N[\dot{G}_{\mathbb{P}}], \mathbb{Q}/\dot{G}_{\mathbb{P}})$ -generic condition. Let G be a V -generic filter on \mathbb{P} with $z \in G$. By Lemma 2.10 applied to z and q_N , we get that $N[G] \cap V \subseteq N$ and q_N is a strongly $(N[\dot{G}_{\mathbb{P}}], \mathbb{Q}/\dot{G}_{\mathbb{P}})$ -generic condition. Finally, if $p \in N[G] \cap (\mathbb{Q}/G)$, then $p \in N \cap \mathbb{Q}$, so p and q_N are compatible in \mathbb{Q} by the universality of q_N . As p and q_N are in \mathbb{Q}/G , they are compatible in \mathbb{Q}/G by Lemma 2.9. \square

Recall the following definition of Mitchell, which plays a prominent role in [9].

Definition 2.12. *A forcing poset \mathbb{Q} is said to have the ω_1 -approximation property if \mathbb{Q} forces that whenever X is a subset of V such that for every set N which is countable in V , $N \cap X$ is in V , then X is in V .*

In other words, \mathbb{Q} has the ω_1 -approximation property iff \mathbb{Q} forces that the pair $(V, V[\dot{G}_{\mathbb{Q}}])$ has the ω_1 -approximation property.

Proposition 2.13. *Let \mathbb{Q} be a forcing poset and assume that \mathbb{Q} is strongly proper on a stationary set. Then \mathbb{Q} satisfies the ω_1 -approximation property.*

Proposition 2.13 is a special case of [8, Lemma 6]. The proof goes roughly as follows. Suppose for a contradiction that a condition p forces that \dot{X} is a counterexample to the approximation property. Fix N an elementary substructure with $p, \dot{X} \in N$ and $q \leq p$ which is a strongly N -generic condition. Extend q to r which decides the value of $N \cap \dot{X}$. Then $r \restriction N$ has extensions v and w in N which disagree

on whether a certain set in N is in \dot{X} . Since r decides the value of $N \cap \dot{X}$, it cannot be compatible with both v and w , giving a contradiction.

As an immediate consequence of Theorem 2.11 and Proposition 2.13, we get the following corollary.

Corollary 2.14. *Let \mathbb{Q} be a forcing poset which has greatest lower bounds, and assume that \mathbb{Q} has universal strongly generic conditions on a stationary set. Let \mathbb{P} be a regular suborder of \mathbb{Q} satisfying $\ast(\mathbb{P}, \mathbb{Q})$. Then \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ satisfies the ω_1 -approximation property.*

3. PRODUCTS

We will show that some of the properties studied in the previous section are preserved under products.¹ This information will be used in our consistency result on $\text{ISP}(\omega_2)$.

Note that if \mathbb{P} and \mathbb{Q} have greatest lower bounds, then so does $\mathbb{P} \times \mathbb{Q}$. Namely, $(p_0, q_0) \wedge (p_1, q_1) = (p_0 \wedge p_1, q_0 \wedge q_1)$.

Lemma 3.1. *Suppose that \mathbb{P} and \mathbb{Q} are forcing posets with greatest lower bounds which both satisfy the property that whenever x , y , and z are pairwise compatible conditions, then x is compatible with $y \wedge z$. Then $\mathbb{P} \times \mathbb{Q}$ satisfies this property as well.*

The proof is straightforward.

Lemma 3.2. *Let \mathbb{P} and \mathbb{Q} be forcing posets with greatest lower bounds. Let λ be a cardinal such that $\lambda_{\mathbb{P}}, \lambda_{\mathbb{Q}} \leq \lambda$ and let S be a subset of $P_{\omega_1}(H(\lambda))$. Suppose that \mathbb{P} and \mathbb{Q} both have universal strongly generic conditions on S . Then $\mathbb{P} \times \mathbb{Q}$ has universal strongly generic conditions on S .*

Proof. Let N be in S . Fix a universal strongly (N, \mathbb{P}) -generic condition p and a universal strongly (N, \mathbb{Q}) -generic condition q . We claim that (p, q) is a universal strongly $(N, \mathbb{P} \times \mathbb{Q})$ -generic condition.

So let (u, v) be in $N \cap (\mathbb{P} \times \mathbb{Q})$. Then by universality, p and u are compatible in \mathbb{P} , and q and v are compatible in \mathbb{Q} . Hence $(p \wedge u, q \wedge v) \leq (p, q), (u, v)$. So (p, q) and (u, v) are compatible in $\mathbb{P} \times \mathbb{Q}$.

Let $r \mapsto r \restriction N$ be a map from conditions $r \leq p$ in \mathbb{P} to $N \cap \mathbb{P}$ witnessing that p is a strongly (N, \mathbb{P}) -generic condition. Similarly, let $s \mapsto s \restriction N$ be a map from conditions $s \leq q$ in \mathbb{Q} to $N \cap \mathbb{Q}$ witnessing that q is a strongly (N, \mathbb{Q}) -generic condition.

For $(r, s) \leq (p, q)$ in $\mathbb{P} \times \mathbb{Q}$, define $(r, s) \restriction N = (r \restriction N, s \restriction N)$, which is clearly in $N \cap (\mathbb{P} \times \mathbb{Q})$. Assume that $(y, z) \leq (r \restriction N, s \restriction N)$ is in $N \cap (\mathbb{P} \times \mathbb{Q})$. Then y is compatible with r in \mathbb{P} and z is compatible with s in \mathbb{Q} . So $(r \wedge y, s \wedge z)$ is in $\mathbb{P} \times \mathbb{Q}$ and is below (r, s) and (y, z) . So every extension of $(r, s) \restriction N$ in $N \cap (\mathbb{P} \times \mathbb{Q})$ is compatible with (r, s) . \square

A similar argument shows that if \mathbb{P} and \mathbb{Q} are strongly proper, then so is $\mathbb{P} \times \mathbb{Q}$. In other words, strong properness is productive. This is in contrast to properness, which is not productive; see [12, Chapter XVII, 2.12].

¹Similar results were obtained previously by Friedman [2, Lemma 3] and Neeman [11, Claim 3.8].

4. VARIATIONS

We consider two variations of Theorem 2.11 on quotients of strongly proper forcings. First, we discuss factoring a forcing poset over an elementary substructure below a condition. Secondly, we introduce a weakening of strongly proper which is called non-diagonally strongly proper, and show that this property is sometimes preserved under taking quotients.

Definition 4.1. *Let \mathbb{P} and \mathbb{Q} be forcing posets. A function $f : \mathbb{P} \rightarrow \mathbb{Q}$ is a regular embedding if:*

- (1) *for all p and q in \mathbb{P} , $q \leq p$ implies $f(q) \leq f(p)$;*
- (2) *for all p and q in \mathbb{P} , if $f(p)$ and $f(q)$ are compatible in \mathbb{Q} , then p and q are compatible in \mathbb{P} ;*
- (3) *if A is a maximal antichain of \mathbb{P} , then $f[A]$ is a maximal antichain of \mathbb{Q} .*

It is straightforward to show that if $f : \mathbb{P} \rightarrow \mathbb{Q}$ is a regular embedding, then $f[\mathbb{P}]$ is a regular suborder of \mathbb{Q} .

Previously we have focused on strongly generic conditions for countable models. The next lemma and theorem are useful when the model under consideration is uncountable.

Recall that $\mathbb{P}/q = \{r \in \mathbb{P} : r \leq q\}$, where \mathbb{P} is a forcing poset and $q \in \mathbb{P}$.

Lemma 4.2. *Let \mathbb{P} be a forcing poset with greatest lower bounds, $\chi \geq \lambda_{\mathbb{P}}$ a regular cardinal, and $N \prec (H(\chi), \in, \mathbb{P})$. Suppose that q is a universal strongly N -generic condition. Applying the universality of q , define a function $f : (N \cap \mathbb{P}) \rightarrow (\mathbb{P}/q)$ by $f(p) = q \wedge p$.*

Then f is a regular embedding. Moreover, for any V -generic filter G on $N \cap \mathbb{P}$, for all $r \in \mathbb{P}/q$, $r \in (\mathbb{P}/q)/f[G]$ iff r is compatible with every condition in G .

Proof. We prove first that f is a regular embedding. If $t \leq s$ in $N \cap \mathbb{P}$, then easily $q \wedge t \leq q \wedge s$. Now let s and t be in $N \cap \mathbb{P}$ and assume that $w \leq q \wedge s, q \wedge t$. Then $w \leq s, t$. By the elementarity of N , there is a condition w' in $N \cap \mathbb{P}$ such that $w' \leq s, t$.

Now let A be a maximal antichain of $N \cap \mathbb{P}$, and we will show that $f[A]$ is predense below q . Since f preserves incompatibility, it follows that $f[A]$ is a maximal antichain of \mathbb{P}/q . Let D be the set of $p \in N \cap \mathbb{P}$ for which there is $s \in A$ such that $p \leq s$. Then D is dense in $N \cap \mathbb{P}$. Since q is strongly N -generic, D is predense below q . Consider $r \in \mathbb{P}/q$. Then there is $u \in D$ and t such that $t \leq r, u$. By the definition of D , there is $s \in A$ such that $u \leq s$. Then $t \leq r, s$. As $t \leq q$, we have that $t = q \wedge t \leq q \wedge s = f(s)$. Hence r is compatible with $f(s)$, and $f(s) \in f[A]$.

Let G be a V -generic filter on $N \cap \mathbb{P}$. Let $r \leq q$ be given. Assume that $r \in (\mathbb{P}/q)/f[G]$, which means that r is compatible with every condition in $f[G]$. Consider $s \in G$. Fix $t \leq r, f(s)$. Since $f(s) = q \wedge s \leq s$, it follows that $t \leq r, s$. So r and s are compatible. Conversely, assume that $r \leq q$ and r is compatible with every condition in G . Let $f(s)$ in $f[G]$ be given, where $s \in G$. Since $s \in G$, r and s are compatible, so fix $t \leq r, s$. Then $t \leq q$, so $t \leq q \wedge s = f(s)$. Hence r and $f(s)$ are compatible. \square

For a V -generic filter G on $N \cap \mathbb{P}$, let $(\mathbb{P}/q)/G$ denote the forcing poset consisting of conditions in \mathbb{P}/q which are compatible with every condition in G . By the lemma,

$$(\mathbb{P}/q)/G = (\mathbb{P}/q)/f[G].$$

Since $N \cap \mathbb{P}$ is forcing equivalent to $f[N \cap \mathbb{P}]$, it follows that \mathbb{P}/q is forcing equivalent to the two-step iteration

$$(N \cap \mathbb{P}) * ((\mathbb{P}/q)/\dot{G}_{N \cap \mathbb{P}}).$$

If H is a V -generic filter on \mathbb{P} which contains q , then by Lemma 1.6(1), $H \cap f[N \cap \mathbb{P}]$ is a V -generic filter on $f[N \cap \mathbb{P}]$. Since f is a dense embedding, $f^{-1}[H \cap f[N \cap \mathbb{P}]]$ is a V -generic filter on $N \cap \mathbb{P}$. But this latter set is just equal to $H \cap N$. For if $f(p) = q \wedge p$ is in H , then since $q \wedge p \leq p$, p is in H . And if $p \in H \cap N$, then since $q \in H$, $f(p) = q \wedge p \in H$. By Lemma 1.6(1), it follows that $H \cap N$ is a V -generic filter on $N \cap \mathbb{P}$, $H_q := \{r \in H : r \leq q\}$ is a $V[H \cap N]$ -generic filter on $(\mathbb{P}/q)/(H \cap N)$, and $V[H] = V[H \cap N][H_q]$.

Theorem 4.3. *Let \mathbb{P} be a forcing poset with greatest lower bounds satisfying $*(\mathbb{P}, \mathbb{P})$, $\chi \geq \lambda_{\mathbb{P}}$ a regular cardinal, and $N \prec (H(\chi), \in, \mathbb{P})$. Suppose that \mathbb{P} has universal strongly generic conditions on a stationary set, and q is a universal strongly N -generic condition.*

Then the forcing poset $N \cap \mathbb{P}$ forces that the quotient $(\mathbb{P}/q)/\dot{G}_{N \cap \mathbb{P}}$ has universal strongly generic conditions on a stationary set. Moreover, whenever H is a V -generic filter on \mathbb{P} which contains q , we have that $V[H] = V[H \cap N][K]$, where $H \cap N$ is a V -generic filter on $N \cap \mathbb{P}$, K is a $V[H \cap N]$ -generic filter on $(\mathbb{P}/q)/(H \cap N)$, and the pair $(V[H \cap N], V[H])$ satisfies the ω_1 -approximation property.

Proof. We claim that the forcing poset $N \cap \mathbb{P}$ forces that the quotient $(\mathbb{P}/q)/\dot{G}_{N \cap \mathbb{P}}$ has universal strongly generic conditions on a stationary set. In particular, $N \cap \mathbb{P}$ forces that $(\mathbb{P}/q)/\dot{G}_{N \cap \mathbb{P}}$ is strongly proper on a stationary set, and hence has the ω_1 -approximation property.

Let $f : N \cap \mathbb{P} \rightarrow \mathbb{P}/q$ be the function $f(p) = q \wedge p$. By Theorem 2.11, it suffices to show that the regular suborder $f[N \cap \mathbb{P}]$ of \mathbb{P}/q satisfies property $*(f[N \cap \mathbb{P}], \mathbb{P}/q)$. But this follows immediately from the fact that \mathbb{P} satisfies $*(\mathbb{P}, \mathbb{P})$.

The second conclusion of the theorem follows from this claim together with the analysis given prior to the statement of the theorem. \square

An example of factoring a forcing poset over an uncountable elementary substructure appears in the final argument of Mitchell's theorem on the approachability ideal in [9, Section 3]. Mitchell uses the existence of what he calls tidy strongly generic conditions to show that the quotient has the ω_1 -approximation property. Theorem 4.3 provides a different justification for Mitchell's argument which avoids tidy conditions.

For our second variation of Theorem 2.11, we introduce a weakening of strong properness which is useful in situations where we desire quotients to have the ω_1 -approximation property but universal conditions do not exist.

Definition 4.4. *A forcing poset \mathbb{Q} is non-diagonally strongly proper if for every condition p in \mathbb{Q} , there are stationarily many N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that $p \in N$ and there exists $q \leq p$ such that q is a strongly N -generic condition.*

As usual, if we replace $\lambda_{\mathbb{Q}}$ with any cardinal $\chi \geq \lambda_{\mathbb{Q}}$ in the definition we get an equivalent statement.

Proposition 4.5. *Let \mathbb{Q} be a non-diagonally strongly proper forcing poset. Then \mathbb{Q} satisfies the ω_1 -approximation property.*

The proof is the same as the one sketched after Proposition 2.13.

Theorem 4.6. *Suppose that \mathbb{Q} is a forcing poset with greatest lower bounds satisfying that for every condition p in \mathbb{Q} , there are stationarily many N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that $p \in N$ and there exists a strongly N -generic condition extending p . Let \mathbb{P} be a regular suborder of \mathbb{Q} satisfying property $\ast(\mathbb{P}, \mathbb{Q})$. Then \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ is non-diagonally strongly proper. In particular, \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ has the ω_1 -approximation property.*

Proof. Let χ be a regular cardinal such that $\mathbb{Q} \in H(\chi)$. Then \mathbb{Q} forces that χ is regular and $\lambda_{\mathbb{Q}/\dot{G}_{\mathbb{P}}} \leq \chi$. Let \dot{F} be a \mathbb{P} -name for a function $\dot{F} : (H(\chi)^{V[\dot{G}_{\mathbb{P}}]})^{<\omega} \rightarrow H(\chi)^{V[\dot{G}_{\mathbb{P}}]}$, let $s \in \mathbb{P}$, and suppose that s forces that \dot{p} is a condition in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. We will find an extension of s in \mathbb{P} which forces that there exists a countable set $M \subseteq H(\chi)^{V[\dot{G}_{\mathbb{P}}]}$ which is closed under \dot{F} , contains \dot{p} , and there is a strongly $(M, \mathbb{Q}/\dot{G}_{\mathbb{P}})$ -generic condition below \dot{p} .

Extending s if necessary, assume that for some $p \in \mathbb{Q}$, s forces that $\dot{p} = \check{p}$. Since s forces that \dot{p} is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$, s forces that \dot{p} is compatible with every condition in $\dot{G}_{\mathbb{P}}$. In particular, s forces that \dot{p} is compatible with s . So p and s are compatible.

Let $H : H(\chi)^{<\omega} \rightarrow H(\chi)$ be a function such that \mathbb{P} forces that $H(\dot{a}_0, \dots, \dot{a}_n) = \dot{F}(\dot{a}_0, \dots, \dot{a}_n)$ for all \mathbb{P} -names $\dot{a}_0, \dots, \dot{a}_n$ in $H(\chi)$. Since \mathbb{Q} is non-diagonally strongly proper, fix a countable set $N \prec H(\chi)$ closed under H such that \mathbb{P} , \mathbb{Q} , s , and p are in N and there is $q \leq p \wedge s$ which is a strongly (N, \mathbb{Q}) -generic condition.

As \mathbb{P} is a regular suborder of \mathbb{Q} , fix z in \mathbb{P} such that every extension of z in \mathbb{P} is compatible with q . Then z forces that q is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. Since z is compatible with q in \mathbb{Q} and $q \leq s$, z is compatible with s in \mathbb{Q} . Since \mathbb{P} is a regular suborder and z and s are in \mathbb{P} , z and s are compatible in \mathbb{P} . So without loss of generality assume that $z \leq s$.

Since N is closed under H , \mathbb{P} forces that $N[\dot{G}_{\mathbb{P}}]$ is closed under \dot{F} . By Lemma 2.10, z forces that q is a strongly $(N[\dot{G}_{\mathbb{P}}], \mathbb{Q}/\dot{G}_{\mathbb{P}})$ -generic condition. Also $z \leq s$, and since $p \in N$, z forces that $\dot{p} = \check{p}$ is in $N[\dot{G}_{\mathbb{P}}]$. \square

We give an example of a non-diagonally strongly proper forcing poset which is not strongly proper on a stationary set.

Consider a stationary set $S \subseteq \omega_1$. Recall the forcing poset \mathbb{P}_S for adding a club subset of S with finite conditions ([1, Theorem 3]). A condition in \mathbb{P}_S is a finite set p of ordered pairs $\langle \alpha, \beta \rangle$ such that $\alpha \in S$ and $\alpha \leq \beta < \omega_1$, and whenever $\langle \alpha, \beta \rangle$ and $\langle \alpha', \beta' \rangle$ are in p then it is not the case that $\alpha < \alpha' \leq \beta$. Let $q \leq p$ in \mathbb{P}_S if $p \subseteq q$.

Proposition 4.7. *Let N be a countable elementary substructure of $(H(\omega_1), \in, S)$ such that $N \cap \omega_1 \in S$, and let $p \in N \cap \mathbb{P}_S$. Then $p \cup \{ \langle N \cap \omega_1, N \cap \omega_1 \rangle \}$ is a strongly (N, \mathbb{P}_S) -generic condition.*

The proof is straightforward. The forcing poset \mathbb{P}_S preserves all cardinals and adds a club subset of S ; see [1] for the details.

Now consider S_0 and S_1 which are disjoint stationary subsets of ω_1 . Define \mathbb{Q} as the forcing poset consisting of conditions of the form (i, p) , where $i \in \{0, 1\}$ and $p \in \mathbb{P}_{S_i}$. Let $(j, q) \leq (i, p)$ if $i = j$ and $q \leq p$ in \mathbb{P}_{S_i} .

Proposition 4.8. *The forcing poset \mathbb{Q} is non-diagonally strongly proper.*

Proof. Let (i, p) be a condition in \mathbb{Q} and let $F : H(\omega_1)^{<\omega} \rightarrow H(\omega_1)$ be a function. Since S_i is stationary, we can fix a countable elementary substructure N

of $(H(\omega_1), \in, F, S_0, S_1)$ such that $N \cap \omega_1 \in S_i$ and $(i, p) \in N$. Then Proposition 4.7 implies that $(i, p \cup \{ \langle N \cap \omega_1, N \cap \omega_1 \rangle \})$ is a strongly (N, \mathbb{Q}) -generic condition extending (i, p) . \square

Proposition 4.9. *The forcing poset \mathbb{Q} is not strongly proper on a stationary set.*

Proof. Let \dot{C} be a \mathbb{Q} -name for the club subset of ω_1 added by \mathbb{Q} . Suppose that N is an elementary substructure of $H(\chi)$ for some regular $\chi > \omega_1$ such that S_0, S_1, \mathbb{Q} , and \dot{C} are in N . Since S_0 and S_1 are disjoint, $N \cap \omega_1$ cannot be in both S_0 and S_1 . Fix $j \in \{0, 1\}$ such that $N \cap \omega_1$ is not in S_j . We claim that the condition (j, \emptyset) , which is in $N \cap \mathbb{Q}$, does not have a strongly N -generic extension. In fact, it does not have an N -generic extension.

Suppose for a contradiction that $(j, q) \leq (j, \emptyset)$ is N -generic. Then (j, q) forces that $N[\dot{G}] \cap V = N$, and in particular, $N[\dot{G}] \cap \omega_1 = N \cap \omega_1$. Since $\dot{C} \in N$, \mathbb{Q} forces that $N[\dot{G}] \cap \omega_1 \in \dot{C}$. So (j, q) forces that $N \cap \omega_1 \in \dot{C}$. Since (j, q) forces that \dot{C} is a subset of S_j , (j, q) forces that $N \cap \omega_1 \in S_j$. But $N \cap \omega_1$ is not in S_j , so we have a contradiction. \square

5. A COUNTEREXAMPLE

We give an example of a strongly proper forcing poset \mathbb{Q} and a regular suborder \mathbb{P} of \mathbb{Q} such that \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ is not strongly proper on a stationary set. We will make use of the following well known fact.

Lemma 5.1. *If \mathbb{Q} is strongly proper on a stationary set, then any generic extension of V by \mathbb{Q} contains a V -generic Cohen real.*

Proof. (Sketch) Let G be a V -generic filter on \mathbb{Q} . By a density argument, there exists a countable elementary substructure N with $\mathbb{Q} \in N$ and a strongly N -generic condition $q \in G$. Then $N \cap G$ is a V -generic filter on $N \cap \mathbb{Q}$. But $N \cap \mathbb{Q}$ is a countable nontrivial forcing poset, and hence is forcing equivalent to Cohen forcing $\text{ADD}(\omega)$. \square

So it suffices to define a strongly proper forcing poset \mathbb{Q} and a regular suborder \mathbb{P} of \mathbb{Q} such that \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ is nontrivial and does not add reals over $V[\dot{G}_{\mathbb{P}}]$.

Assume that $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Let \mathcal{X} denote the set of all countable elementary substructures of $H(\omega_3)$. The forcing poset \mathbb{Q} consists of finite coherent adequate subsets of \mathcal{X} , ordered by inclusion. The definition of coherent adequate is beyond the scope of the paper, although we give additional information about adequacy in the next section. Roughly speaking, a coherent adequate set of models satisfies that any two models in it are membership comparable up to some initial segment of the models, and if the models are equal up to some initial segment, then they are isomorphic. For the complete definition, see [6, Section 1].

The following lists the properties of the forcing poset \mathbb{Q} which we will use. For a proof of (1) and (2) see [6]. For a proof of (3) see [10].

Proposition 5.2. *The following statements hold:*

- (1) \mathbb{Q} is strongly proper and ω_2 -c.c.
- (2) \mathbb{Q} forces CH .
- (3) \mathbb{Q} forces that there exists an ω_1 -Kurepa tree with ω_3 many distinct branches.

Since \mathbb{Q} does not have greatest lower bounds, we will consider the Boolean completion of \mathbb{Q} , which we denote by \mathcal{B} . Then easily \mathcal{B} has the ω_2 -c.c., preserves all cardinals, and forces CH. Also a straightforward argument shows that since \mathbb{Q} is strongly proper, so is \mathcal{B} .

Let us find a regular suborder \mathbb{P} of \mathcal{B} which forces that $\mathcal{B}/\dot{G}_{\mathbb{P}}$ is nontrivial and does not add reals over $V[\dot{G}_{\mathbb{P}}]$. Since \mathcal{B} forces CH, we can fix a sequence $\langle \dot{r}_i : i < \omega_1 \rangle$ of \mathcal{B} -names for subsets of ω such that \mathcal{B} forces that every subset of ω is equal to \dot{r}_i for some $i < \omega_1$. Moreover, we assume that each name \dot{r}_i is a nice \mathcal{B} -name given by a sequence of antichains $\langle A_n^i : n < \omega \rangle$. So (p, \dot{n}) is in the name \dot{r}_i iff $p \in A_n^i$. Since \mathcal{B} is ω_2 -c.c., each antichain A_n^i has size at most ω_1 .

Fix a regular cardinal $\chi > \omega_3$ such that \mathbb{Q} , \mathcal{B} , $\langle \dot{r}_i : i < \omega_1 \rangle$, and $\langle A_n^i : n < \omega \rangle$ for all $i < \omega_1$ are members of $H(\chi)$. Let N be an elementary substructure of $H(\chi)$ containing these parameters such that N has size ω_2 and $N^{\omega_1} \subseteq N$. This is possible since we are assuming that $2^{\omega_1} = \omega_2$. Now let $\mathbb{P} := N \cap \mathcal{B}$.

Proposition 5.3. *The forcing poset \mathbb{P} is a regular suborder of \mathcal{B} which forces that $\mathcal{B}/\dot{G}_{\mathbb{P}}$ is a nontrivial forcing which does not add reals.*

Proof. Suppose that p and q are in \mathbb{P} and are compatible in \mathcal{B} . Then p and q are in N , so by elementarity, $p \wedge q$ is in $N \cap \mathcal{B} = \mathbb{P}$. So they are compatible in \mathbb{P} . Now let A be a maximal antichain of \mathbb{P} . Then A is an antichain of \mathcal{B} and is a subset of $\mathcal{B} \cap N$. Since \mathcal{B} is ω_2 -c.c., A has size at most ω_1 . But $N^{\omega_1} \subseteq N$, so $A \in N$. Suppose for a contradiction that A is not a maximal antichain of \mathcal{B} . Then there is a condition in \mathcal{B} which is incompatible with every member of A . Since $A \in N$, by elementarity there is a condition in $N \cap \mathcal{B} = \mathbb{P}$ which is incompatible with every member of A . But then A is not maximal in \mathbb{P} , which is a contradiction.

The forcing poset \mathbb{P} has size ω_2 and is ω_2 -c.c. Since $2^{\omega_1} = \omega_2$ in the ground model, \mathbb{P} forces that $2^{\omega_1} = \omega_2$ by a standard nice name argument. But \mathcal{B} forces that there is an ω_1 -tree with ω_3 many distinct branches, and hence $2^{\omega_1} \geq \omega_3$. It follows that \mathbb{P} forces that $\mathcal{B}/\dot{G}_{\mathbb{P}}$ is a nontrivial forcing.

To show that \mathbb{P} forces that $\mathcal{B}/\dot{G}_{\mathbb{P}}$ does not add reals, let G be a V -generic filter on \mathcal{B} . Let $G_{\mathbb{P}} := G \cap \mathbb{P}$, which is a V -generic filter on \mathbb{P} . Let r be a subset of ω in $V[G]$, and we will show that r is in $V[G_{\mathbb{P}}]$. By assumption, there is $i < \omega_1$ such that $r = \dot{r}_i$, where $\dot{r}_i := \dot{r}_i^G$. For each $n < \omega$, $n \in r_i$ iff $A_n^i \cap G \neq \emptyset$. But A_n^i is in N and hence is a subset of N since A_n^i has size at most ω_1 . So $n \in r_i$ iff $A_n^i \cap G \cap N \neq \emptyset$ iff $A_n^i \cap G_{\mathbb{P}} \neq \emptyset$. It follows that r_i , and hence r , is in $V[G_{\mathbb{P}}]$. \square

We note that in fact \mathbb{P} forces that $\mathcal{B}/\dot{G}_{\mathbb{P}}$ does not have the ω_1 -approximation property. Recall that \mathbb{Q} forces that there exists an ω_1 -tree with ω_3 many distinct branches. By looking at the details of the definition of this tree as presented in [10], one can argue that this tree already exists in a generic extension by \mathbb{P} . Since $2^{\omega_1} = \omega_2$ in the generic extension by \mathbb{P} , \mathbb{P} forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ adds new branches to the tree. But any initial segment of such a branch lies in the extension by \mathbb{P} , and this easily implies that the quotient does not have the ω_1 -approximation property.

6. A STRONGLY PROPER COLLAPSE

We now turn towards proving our consistency result. We will construct a model in which $\text{ISP}(\omega_2)$ holds and the continuum is greater than ω_2 . The forcing poset used to obtain this model will be of the form $\mathbb{P} \times \text{ADD}(\omega, \lambda)$, where \mathbb{P} is a strongly proper

forcing which collapses a supercompact cardinal κ to become ω_2 , and $\text{ADD}(\omega, \lambda)$ is the forcing which adds λ many Cohen reals.

In this section we will describe the forcing poset \mathbb{P} . Essentially this forcing is the pure side condition forcing consisting of finite adequate subsets of κ ordered by inclusion, where the notion of adequate set is the same as that defined by Krueger in [3], except that ω_2 is replaced by a strongly inaccessible cardinal κ .²

Assume for the rest of the section that κ is a strongly inaccessible cardinal. In particular, $H(\kappa)$ has size κ . Fix a bijection $\pi : \kappa \rightarrow H(\kappa)$. Then the structure $(H(\kappa), \in, \pi)$ has definable Skolem functions. For any set $a \subseteq \kappa$, let $Sk(a)$ denote the closure of a under some (or equivalently any) set of definable Skolem functions.

Let C be the club set of $\alpha < \kappa$ such that $Sk(\alpha) \cap \kappa = \alpha$. Let Λ be the set of cardinals $\beta < \kappa$ such that β is a limit point of C with uncountable cofinality, and for all $\gamma < \beta$, $\gamma^\omega < \beta$. Note that Λ is stationary in κ . Let \mathcal{X} be the set of countable subsets a of κ such that $Sk(a) \cap \kappa = a$ and for all $\gamma \in a$, $\sup(C \cap \gamma) \in a$.

Using κ in place of ω_2 and the sets C , Λ , and \mathcal{X} just described, it is possible to develop the basic ideas of adequate sets word for word as in [3]. We will give an overview some of the main points. The interested reader is invited to read Sections 1–4 of [3] for the complete details.

For a set $M \in \mathcal{X}$, define Λ_M as the set of $\beta \in \Lambda$ such that

$$\beta = \min(\Lambda \setminus \sup(M \cap \beta)).$$

By Lemma 2.4 of [3], for all M and N in \mathcal{X} , $\Lambda_M \cap \Lambda_N$ has a largest element. This largest element is defined as $\beta_{M,N}$, the *comparison point of M and N* .

One of the most important properties of the comparison point of M and N is expressed in the following inclusion:

$$(M \cup \lim(M)) \cap (N \cup \lim(N)) \subseteq \beta_{M,N}.$$

This is proved in [3, Proposition 2.6].

Definition 6.1. *A finite set $A \subseteq \mathcal{X}$ is adequate if for all M and N in A , either $M \cap \beta_{M,N} \in Sk(N)$, $N \cap \beta_{M,N} \in Sk(M)$, or $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.*

Note that A is adequate if for all M and N in A , $\{M, N\}$ is adequate. Since $\omega_1 \leq \beta_{M,N}$, if $\{M, N\}$ is adequate then $M \cap \beta_{M,N} \in Sk(N)$ iff $M \cap \omega_1 \in N$, and $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ iff $M \cap \omega_1 = N \cap \omega_1$.

Proposition 6.2. *If A is adequate, $M \in \mathcal{X}$, and $A \in Sk(M)$, then $A \cup \{M\}$ is adequate.*

See [3, Proposition 3.5].

If $N \in \mathcal{X}$ and $\beta \in \Lambda$, then $N \cap \beta$ is in \mathcal{X} ([3, Lemma 1.10]). Let us say that an adequate set A is *N -closed* if for all $M \in A$, $M \cap \beta_{M,N} \in A$.

Proposition 6.3. *Let A be adequate and $N \in A$. Then there exists an adequate set B such that $A \subseteq B$ and B is N -closed.*

This follows from [3, Proposition 3.4]. The next result appears as [3, Proposition 3.9].

Proposition 6.4. *Let A be adequate, $N \in A$, and suppose that A is N -closed. Let B be adequate such that $A \cap Sk(N) \subseteq B \subseteq Sk(N)$. Then $A \cup B$ is adequate.*

²Mitchell [9] was the first to define a strongly proper collapse of an inaccessible to become ω_2 . Neeman [11] gives another example using his method of sequences of models of two types.

Now we are ready to define the forcing poset \mathbb{P} . Let \mathbb{P} consist of conditions A such that A is a finite adequate subset of \mathcal{X} . Let $B \leq A$ in \mathbb{P} if $A \subseteq B$.

Lemma 6.5. *The forcing poset \mathbb{P} has greatest lower bounds. Namely, for compatible conditions A and B in \mathbb{P} , $A \wedge B$ equals $A \cup B$.*

Proof. Suppose that A and B are compatible, and let $C \leq A, B$. Since the ordering of \mathbb{P} is inclusion, $A \cup B \subseteq C$. For all M and N in $A \cup B$, $\{M, N\}$ is adequate since they are in C . So $A \cup B$ is adequate. Hence $A \cup B \in \mathbb{P}$ and $C \leq A \cup B$. \square

Lemma 6.6. *For all A, B , and C in \mathbb{P} which are pairwise compatible, A is compatible with $B \cup C$.*

Proof. Let M and N be in $A \cup (B \cup C)$. Then M and N are either both in $A \cup B$, $A \cup C$, or $B \cup C$. Since each of these three sets is adequate, $\{M, N\}$ is adequate. \square

Proposition 6.7. *The forcing poset \mathbb{P} has universal strongly generic conditions on a club.*

Proof. Note that \mathcal{X} is a club subset of $P_{\omega_1}(\kappa)$. Clearly $\lambda_{\mathbb{P}} = \kappa$. Consider any set N in $P_{\omega_1}(H(\kappa))$ such that $N \cap \kappa \in \mathcal{X}$. We claim that $\{N \cap \kappa\}$ is a universal strongly N -generic condition.

If A is a finite adequate set in $N = Sk(N \cap \kappa)$, then by Proposition 6.2, $A \cup \{N \cap \kappa\}$ is adequate and hence is in \mathbb{P} . Therefore A and $\{N \cap \kappa\}$ are compatible.

To show that $\{N \cap \kappa\}$ is a strongly N -generic condition, let E be the set of B in \mathbb{P} such that $\{N \cap \kappa\} \subseteq B$ and B is N -closed. By Proposition 6.3, E is dense below $\{N \cap \kappa\}$. Define $C \mapsto C \upharpoonright N$ for $C \in E$ by letting $C \upharpoonright N := C \cap N$. Clearly $C \cap N$ is adequate and hence is in \mathbb{P} , $C \cap N \in N$, and $C \leq C \cap N$. If B is a condition in N extending $C \upharpoonright N$, then Proposition 6.4 implies that $B \cup C$ is a condition in \mathbb{P} below B and C , showing that C and B are compatible. \square

Proposition 6.8. *The forcing poset \mathbb{Q} is κ -c.c.*

The proof of this proposition is identical to the proof of [3, Proposition 4.4], so we omit it.

Lemma 6.9. *The forcing poset \mathbb{P} forces that κ equals ω_2 .*

Proof. Since \mathbb{P} preserves ω_1 and κ by the preceding propositions, it suffices to show that for any regular cardinal μ such that $\omega_1 < \mu < \kappa$, \mathbb{P} collapses μ to have size ω_1 . So let such a μ be given and let G be a V -generic filter on \mathbb{P} .

Given any condition A and any ordinal $\gamma < \kappa$, we can fix M in \mathcal{X} such that A , γ , and μ are in $Sk(M)$. Then $A \cup \{M\}$ is a condition in \mathbb{P} by Proposition 6.2. It follows by a density argument that the set

$$F = \{M : \exists A \in G (M \in A \wedge \mu \in M)\}$$

has union equal to κ . In particular, $\bigcup \{M \cap \mu : M \in F\} = \mu$.

Now for any M and N in F , $\mu \in M \cap N$ implies that $\mu < \beta_{M,N}$. Fix A in G such that M and N are in A . Since A is adequate and $\mu < \beta_{M,N}$, $M \cap \mu$ and $N \cap \mu$ are either equal or one is a proper subset of the other. Moreover, since $M \cap \beta_{M,N} \subseteq N$ iff $M \cap \omega_1 \leq N \cap \omega_1$, it follows that the set $\{M \cap \mu : M \in F\}$ is a chain well-ordered by inclusion. As each set in this chain is countable, it must have length at most ω_1 . So μ is the union of ω_1 many countable sets, which implies that μ is collapsed to have size ω_1 in $V[G]$. \square

This completes our treatment of a strongly proper collapse. We note that some variations are possible. By [5], it is possible to eliminate N -closure which is used in Propositions 6.4 and 6.7, so that the family of models \mathcal{X} can be chosen not to be closed under intersections. An alternative development of these ideas can be made in the context of coherent adequate set forcing as described in [4] and [6], to produce a strongly proper collapse which preserves CH.

We also point out that if κ is a Mahlo cardinal, then the forcing poset \mathbb{P} above forces that there are no special Aronszajn trees on ω_2 , and if κ is weakly compact then \mathbb{P} forces that there are no Aronszajn trees on ω_2 . These facts follow by arguments similar to Mitchell's original proof in [7] using Corollary 2.14. Neeman obtained similar results in [11, Section 5].

7. THE CONSISTENCY RESULT

We start with a model in which GCH holds, κ is a supercompact cardinal, and λ is a cardinal of uncountable cofinality with $\kappa \leq \lambda$. We will define a forcing poset $\mathbb{P} \times \mathbb{Q}$ which collapses κ to become ω_2 , forces $2^\omega = \lambda$, and forces that $\text{ISP}(\omega_2)$ holds.

Let \mathbb{P} be the forcing poset described in the preceding section consisting of finite adequate collections of countable subsets of κ , ordered by inclusion. Let \mathbb{Q} be $\text{ADD}(\omega, \lambda)$. Conditions in \mathbb{Q} are finite partial functions from $\omega \times \lambda$ into 2, ordered by inclusion.

The forcing poset \mathbb{Q} is κ -Knaster. In other words, if $\{q_i : i < \kappa\}$ is a subset of \mathbb{Q} , then there is a cofinal set $X \subseteq \kappa$ such that for all $i < j$ in X , q_i and q_j are compatible. This follows from the Δ -system lemma by a standard argument. Since \mathbb{P} is κ -c.c., it follows easily that $\mathbb{P} \times \mathbb{Q}$ is κ -c.c. Also $\mathbb{P} \times \mathbb{Q}$ has size λ . It follows by a standard argument using nice names that $\mathbb{P} \times \mathbb{Q}$ forces that $2^\omega = \lambda$.

Lemma 7.1. *If p and q are in \mathbb{Q} , then $p \cup q$ is the greatest lower bound of p and q . If p , q , and r are pairwise compatible conditions in \mathbb{Q} , then p is compatible with $q \wedge r$.*

The proof is easy.

Lemma 7.2. *The forcing poset \mathbb{Q} has universal strongly generic conditions on a club.*

Proof. Let N be a countable elementary substructure of $H(\lambda)$. We claim that the empty condition is a universal strongly N -generic condition. Clearly it is compatible with every condition in $N \cap \mathbb{Q}$.

For each $r \in \mathbb{Q}$, let $r \restriction N := r \cap N$. Then $r \restriction N \in N \cap \mathbb{Q}$ and $r \leq r \restriction N$. Suppose that $v \leq r \restriction N$ is in $N \cap \mathbb{Q}$. We claim that r and v are compatible. If $(i, n) \in \text{dom}(r) \cap \text{dom}(v)$, then $(i, n) \in \text{dom}(r) \cap N$. Also $r(i, n) \in N$, so $(i, n, r(i, n)) \in r \cap N$. Since v extends $r \cap N$, $r(i, n) = v(i, n)$. It follows that $r \cup v$ is a condition below r and v . \square

Proposition 7.3. *The forcing poset $\mathbb{P} \times \mathbb{Q}$ satisfies:*

- (1) $\mathbb{P} \times \mathbb{Q}$ has greatest lower bounds;
- (2) for all pairwise compatible conditions x , y , and z in $\mathbb{P} \times \mathbb{Q}$, x is compatible with $y \wedge z$;
- (3) $\mathbb{P} \times \mathbb{Q}$ has universal strongly generic conditions on a club;
- (4) $\mathbb{P} \times \mathbb{Q}$ is strongly proper.

Proof. (1) and (2) follow from Lemmas 3.1, 6.5, 6.6, and 7.1. (3) follows from Lemma 3.2, Proposition 6.7, and Lemma 7.2. \square

Since $\mathbb{P} \times \mathbb{Q}$ is strongly proper it preserves ω_1 , and since it is κ -c.c., it preserves all cardinals greater than or equal to κ . By Lemma 6.9, every cardinal μ with $\omega_1 < \mu < \kappa$ is collapsed to have size ω_1 . It follows that $\mathbb{P} \times \mathbb{Q}$ forces that κ is equal to ω_2 . As noted above, $\mathbb{P} \times \mathbb{Q}$ forces that $2^\omega = \lambda$.

It remains to show that $\mathbb{P} \times \mathbb{Q}$ forces that $\text{ISP}(\omega_2)$ holds. So fix a regular cardinal χ such that $\mathbb{P} \times \mathbb{Q} \in H(\chi)$. Let \dot{F} be a $(\mathbb{P} \times \mathbb{Q})$ -name for a function $\dot{F} : (H(\chi)^{V[\dot{G}_{\mathbb{P} \times \mathbb{Q}}]})^{<\omega} \rightarrow H(\chi)^{V[\dot{G}_{\mathbb{P} \times \mathbb{Q}}]}$. We will prove that $\mathbb{P} \times \mathbb{Q}$ forces that there exists a set N satisfying:

- (1) N is in $P_\kappa(H(\chi))$;
- (2) $N \cap \kappa \in \kappa$;
- (3) $N \prec H(\chi)$;
- (4) N is closed under \dot{F} ;
- (5) N is an ω_1 -guessing model.

Since κ is supercompact, we can fix an elementary embedding $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > |H(\chi)|$ and $M^{|H(\chi)|} \subseteq M$.

Lemma 7.4. *In V , the function $j \upharpoonright \mathbb{P} \times \mathbb{Q}$ is a regular embedding of $\mathbb{P} \times \mathbb{Q}$ into $j(\mathbb{P} \times \mathbb{Q})$. In particular, the forcing poset $j[\mathbb{P} \times \mathbb{Q}]$ is a regular suborder of $j(\mathbb{P} \times \mathbb{Q})$.*

Proof. Properties (1) and (2) of Definition 4.1 follow immediately from the fact that j is an elementary embedding. For (3), let A be a maximal antichain of $\mathbb{P} \times \mathbb{Q}$. Since $\mathbb{P} \times \mathbb{Q}$ is κ -c.c., $|A| < \kappa$. Therefore $j(A) = j[A]$. By elementarity, in M the set $j(A)$ is a maximal antichain of $j(\mathbb{P} \times \mathbb{Q})$. But being a maximal antichain is upwards absolute, so $j(A) = j[A]$ is a maximal antichain of $j(\mathbb{P} \times \mathbb{Q})$. \square

Since being a regular suborder is downwards absolute, $j[\mathbb{P} \times \mathbb{Q}]$ is a regular suborder of $j(\mathbb{P} \times \mathbb{Q})$ in the model M .

Let $G \times H$ be a V -generic filter on $\mathbb{P} \times \mathbb{Q}$. Let $F := \dot{F}^{G \times H}$. We will prove that in $V[G \times H]$, there exists a set N satisfying properties (1)–(5) above.

Since $j \upharpoonright \mathbb{P} \times \mathbb{Q}$ is an isomorphism of $\mathbb{P} \times \mathbb{Q}$ onto $j[\mathbb{P} \times \mathbb{Q}]$, $j[G \times H]$ is a V -generic filter on $j[\mathbb{P} \times \mathbb{Q}]$ and $V[G \times H] = V[j[G \times H]]$. Let K be a $V[G \times H]$ -generic filter on the quotient forcing $j(\mathbb{P} \times \mathbb{Q})/j[G \times H]$. By Lemma 1.6(2), K is a V -generic filter on $j(\mathbb{P} \times \mathbb{Q})$ and $V[G \times H][K] = V[K]$. Also by Lemma 1.6(2), $j[G \times H] = K \cap j[\mathbb{P} \times \mathbb{Q}]$. In particular, $j[G \times H] \subseteq K$. Hence we can extend the elementary embedding j in $V[K]$ to $j : V[G \times H] \rightarrow M[K]$ by letting $j(\dot{a}^{G \times H}) = j(\dot{a})^K$.

By the elementarity of j , it suffices to prove that $M[K]$ models that there exists a set N satisfying:

- (1) N is in $P_{j(\kappa)}(H(j(\chi)))$;
- (2) $N \cap j(\kappa) \in j(\kappa)$;
- (3) $N \prec H(j(\chi))$;
- (4) N is closed under $j(F)$;
- (5) N is an ω_1 -guessing model.

Let $N := j[H(\chi)^{V[G \times H]}]$.

First, we prove that N is in $M[K]$. Since $\mathbb{P} \times \mathbb{Q} \in H(\chi)^V$, $H(\chi)^{V[G \times H]} = H(\chi)^V[G \times H]$. As $M^{|H(\theta)^V|} \subseteq M$, $H(\theta)^V$ and $j \restriction H(\theta)^V$ are in M . By the definition of the extended embedding j , $N = \{j(\dot{a}^{G \times H}) : \dot{a} \in H(\chi)^V\} = \{j(\dot{a})^K : \dot{a} \in H(\chi)^V\}$. Since $H(\chi)^V$, $j \restriction H(\chi)^V$, and K are in $M[K]$, so is N .

Now let us check that N satisfies properties (1)–(5).

(1) Obviously $N \subseteq H(j(\chi))$ in $M[K]$. Let $f : |H(\theta)^V| \rightarrow H(\theta)^V$ be a bijection in V . The surjection $g : |H(\theta)^V| \rightarrow N$ given by $g(\alpha) = j(f(\alpha))^K$ is in $M[K]$. Since $j(\kappa) > |H(\theta)^V|$, in $M[K]$ we have that $N \in P_{j(\kappa)}(H(j(\chi)))$.

(2) As κ is the critical point of j , $N \cap j(\kappa)$ equals κ , which is in $j(\kappa)$.

(3) Let $L : (H(\chi)^{V[G \times H]})^{<\omega} \rightarrow H(\chi)^{V[G \times H]}$ be a Skolem function for the structure $(H(\chi), \in)$. By elementarity, $j(L)$ is a Skolem function in $M[K]$ for the structure $(H(j(\chi)), \in)$. Easily N is closed under $j(L)$. So $N \prec H(j(\chi))$ in $M[K]$.

(4) Since $H(\chi)^{V[G \times H]}$ is obviously closed under F , easily N is closed under $j(F)$.

(5) To show that N is an ω_1 -guessing model in $M[K]$, by Lemma 1.10 it suffices to show that the pair $(\overline{N}, M[K])$ satisfies the ω_1 -approximation property, where \overline{N} is the transitive collapse of N . Since $N = j[H(\chi)^{V[G \times H]}]$ is isomorphic to $H(\chi)^{V[G \times H]}$, which is transitive, it follows that $\overline{N} = H(\chi)^{V[G \times H]}$. As $H(\chi)^V = H(\chi)^M$, $H(\chi)^{V[G \times H]} = H(\chi)^{M[G \times H]}$. So it suffices to show that the pair $(H(\chi)^{M[G \times H]}, M[K])$ satisfies the ω_1 -approximation property.

By Proposition 7.3, the forcing poset $\mathbb{P} \times \mathbb{Q}$ has greatest lower bounds and has universal strongly generic conditions on a stationary set. And for all pairwise compatible conditions x , y , and z in $\mathbb{P} \times \mathbb{Q}$, x is compatible with $y \wedge z$. By elementarity, the same properties are satisfied by $j(\mathbb{P} \times \mathbb{Q})$ in the model M . In particular, by Lemma 7.4 $j[\mathbb{P} \times \mathbb{Q}]$ is a regular suborder of $j(\mathbb{P} \times \mathbb{Q})$ satisfying property $*(j[\mathbb{P} \times \mathbb{Q}], j(\mathbb{P} \times \mathbb{Q}))$. By Corollary 2.14, the forcing poset $j(\mathbb{P} \times \mathbb{Q})/j[G \times H]$ satisfies the ω_1 -approximation property in the model $M[G \times H]$. Hence the pair $(M[G \times H], M[K])$ satisfies the ω_1 -approximation property.

In $M[K]$ let d be a bounded subset of $H(\chi)^{M[G \times H]} \cap On$ such that for any countable set $b \in H(\chi)^{M[G \times H]}$, $b \cap d \in H(\chi)^{M[G \times H]}$. Since $H(\chi)^{M[G \times H]}$ contains all of its countable subsets in $M[G \times H]$ by the regularity of χ , the ω_1 -approximation property of $(M[G \times H], M[K])$ implies that $d \in M[G \times H]$. But $H(\chi)^{M[G \times H]}$ is closed under bounded sets of ordinals, so $d \in H(\chi)^{M[G \times H]}$.

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